Horizon Complementarity and Casimir Violations of the Null Energy Condition

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ABSTRACT

The principle of *horizon complementarity* is an attempt to extend ideas about black hole complementarity to all horizons, including cosmological ones. The idea is that the degrees of freedom necessary to describe the interior of the cosmic horizon of *one* observer in a given universe are in fact sufficient to account for the physics of that entire universe: the remainder is just a set of redundant copies of the interior of a single cosmic horizon. *These copies must be factored out*, just as one has to factor out gauge redundancies to identify the true degrees of freedom in gauge theory. Motivated by the observation that quantum cosmology favours *compactified* negatively curved spatial sections, we propose to use such geometries to implement horizon complementarity for eternal Inflation. We point out that the "effective finiteness" of such universes has important consequences for physics *inside* the observer's horizon: there is a non-local effect, represented by a Casimir energy. We use our proposed interpretation of complementarity to constrain the gravitational Casimir coupling in two very different ways; the result is an explicit prediction for the value of the coupling.

1. Horizon Complementarity Applied to Cosmology

The most fully developed proposed explanation of the value of the cosmological constant [1][2] is based on the Coleman-De Luccia bubble nucleation mechanism [3][4]. Each nucleation can reduce the value of the vacuum energy; it turns out that the range of possible values is a "discretuum" with gaps small enough to account for the value observed in our Universe[1].

Once we have a concrete mechanism for producing a Universe which resembles the one we observe, we should be able to use it to work towards answers to some fundamental questions. In particular, we should ultimately obtain a specific picture of the structure of our Universe in its earliest stages, even before Inflation [10].

The standard picture of this part of cosmic history is as follows. The passage through the bubble wall mediates a change in the set of distinguished cosmological observers, and the internal observers see a classical spacetime which is infinite to the future [if the vacuum energy is still positive — it is then future asymptotically de Sitter] and also [4] in space: the spatial sections are copies of three-dimensional *hyperbolic space*, H^3.

It has long been known that infinite space gives rise to various apparent paradoxes, which continue to elude a generally accepted solution[2]. It is natural to ask: is the standard interpretation of bubble spacetime geometry misleading us? Could it be that bubble worlds are, in some sense, *effectively finite* [14][15] when non-classical effects are taken into account? If so, the most serious difficulties of eternal Inflation would be resolved at a stroke. We shall argue here that, in particular, the question of *spatial* finiteness is relevant, indeed essential, to a full account of the history of the *observable* Universe, whether or not the finiteness is itself in any way directly observable[3].

As is well known, in an asymptotically de Sitter spacetime no single observer can ever receive signals from more than a finite region of any spatial section: there are cosmological horizons. The spatial sections are only infinite in a physical sense if one takes a *global* perspective, that is, if one imagines collating the observations of an infinite family of observers. As is well known, taking such a global point of view leads to serious difficulties in the case of black hole horizons [18], and it is natural to suggest [19][20][21][22] that analogous problems might arise if one tries to do so here.

In this way one is led to the concept of *horizon complementarity*. This is the natural generalization of the well-known principle of black hole complementarity [18] to *all* horizons: one should not expect to be able to give a consistent account of the physics on both sides of *any* horizon. In other words, the degrees of freedom used by *one* observer are sufficient to describe an entire universe, of whatever size. This is the sense in which an infinite world can be "effectively finite".

The simplicity and reasonableness of this proposal, which just excludes all data that can never affect a given observer, are evident[4]. It is, nevertheless, a very radical departure:

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[1]The basic reference is [5]; see for example [6][7][8][9] for a selection of more recent work on this theme.

[2]For clear discussions and lists of references see [11][12][13].

[3]Throughout this work, we assume that the [spatial] finiteness of the Universe is *not* directly observable at the present time [16], or indeed at any time. This is natural if one uses Inflation, as we do here; see however [17] for a recent discussion of relevant observations.

[4]Note that objects outside the bubble can also have a direct effect on conditions *inside* the cosmological horizon: see [23][24][25]. This has no bearing on the question being discussed here.

the claim is that an infinite universe can be described by the degrees of freedom of a finite one. We are entitled to ask how this works in more detail. One would also like to know whether this drastic reduction has any consequences [through some non-local effect] for the earliest history of that part of the Universe which *can* affect the observer.

In this work, we shall propose an approach to answering such questions. The basic idea is that the effective spatial finiteness of the horizon interior can be expressed concretely in terms of the way the hyperbolic space H^3 can be reduced to a compact quotient [26] of the form H^3/Γ, where Γ is one of the many infinite discrete subgroups of O(1,3) that can act freely, isometrically, and properly discontinuously on H^3. This "taking of a quotient" amounts to declaring that the infinite bulk of H^3 represents an infinitely *redundant description* of a particular finite sub-domain Δ, a fundamental domain containing an observer and defined by Γ. To understand this, note that the faces of the fundamental domain are topologically identified in pairs, so that an observer venturing beyond the boundary of Δ can never find anything "new". From the perspective of the fundamental domain, the full hyperbolic space consists of an infinite number of identical copies of Δ, joined along their boundaries in a manner described by the discrete rotations contained in Γ.

We therefore claim that Γ is like a group of *gauge transformations*, which, like any other such group, must be factored out in order to identify the true physical degrees of freedom[5]. When the process is described in this way, it is clear that this is a precise mathematical implementation of the concepts underlying horizon complementarity, as it applies to Coleman-De Luccia bubble universes. Note that spatial sections of the form H^3/Γ are in fact very natural from a string-theoretic point of view: string winding "transforms geometry to topology" exactly when one compactifies H^3 in the way we propose here: see [28][29].

This interpretation of horizon complementarity is suggested by recent discussions of the quantum creation of a universe from "nothing". It appears that quantum-gravitational processes in this case do favour compact spatial sections, and in fact they favour compact *negatively curved* [or perhaps flat] sections. This has been argued with various emphases by Zeldovich and Starobinsky [30], by Coule and Martin [31], and by Linde [32][33]. Linde in particular stresses that in the case of spacetimes produced by quantum creation from "nothing", compact negatively curved spatial sections are likely to be favoured, because in this case there is no barrier through which one must tunnel, and because *finite worlds are easier to create*. We are essentially proposing to adapt some of the lessons learnt in these studies to the Coleman-De Luccia case.

The reduction of H^3 to H^3/Γ gives us a concrete mathematical basis for investigating the physical consequences of horizon complementarity, as applied to the *inflationary* horizons established immediately after the bubble spacetime emerges from the bubble wall. In fact, we will argue that this implementation of horizon complementarity leads to an extremely specific description of the geometry of the bubble spacetime in its earliest stages. The argument runs as follows. As is well known, the compactification of spatial sections leads to a cosmological version of the Casimir effect. In a Friedmann

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[5]The idea that the spatial sections of bubble universes might not be globally identical to H^3 was suggested in [27]; the modification suggested there is however very much more drastic than what we are proposing here.

cosmological model, the Casimir energy is represented by a term in which the Casimir coupling to gravity is described by a certain constant, denoted by α here. We show that the requirement of non-perturbative stability [in string theory] puts an upper bound on α, while horizon complementarity itself puts a direct lower bound on it. The range of α values compatible with both constraints proves to be extremely narrow, thus effectively allowing us to compute α. In this way we can specify the spacetime metric of the earliest [bubble] spacetime quite precisely.

We begin by describing our proposal in greater detail.

2. The Proposal in More Detail

Let us begin by examining the conformal diagram representing bubble nucleation [Figure 1]. Note that bubble nucleation cannot be represented in an entirely classical manner, but the essential points are still captured by the diagram. Nucleation occurs along AB, and the spacetime trajectory of the outer surface of the bubble wall is represented by BD. Following Aguirre and Gratton [34], we think of bubble nucleation as a process which separates the spacetime into three zones: the original spacetime and the bubble interior [ECD in the diagram] can be described more or less accurately in a semi-classical way, but the interior of the bubble wall [ACDB in the diagram] is a predominantly quantum domain. As with any quantum process [like the decay of an unstable particle, as discussed by Aguirre and Gratton], the region ACDB can be regarded as a system which *prepares the initial conditions for the subsequent semi-classical evolution.* We do not know how to describe the spacetime geometry inside ACDB; but the spacelike surface CD must still bear the imprint of the non-classical physics [such as that which gives rise to horizon complementarity] which govern the interior of the bubble wall. The semi-classical spacetime inside the bubble has to be *cut off* along CD; but this will not be done in an arbitrary way: it will be done in a manner compatible with whatever we know about the properties of the strictly quantum domain ACDB.

We propose that horizon complementarity can be formulated in the following way: we think of CD as being broken up into an infinite collection of identical copies of a fixed finite *fundamental domain* Δ [26], and declare that the physical degrees of freedom associated with Δ suffice to describe CD completely. If we think of Δ as being centred on the origin of coordinates, its boundary is symbolised by the heavy dot on CD in Figure 1. [Other possible locations for Δ can be pictured as intervals on CD, which would appear to become smaller and smaller as D is approached. Note that this way of representing the boundary of Δ is schematic, in the sense that the distance to the boundary depends both on the choice of the origin and on direction.] Because the entire bubble spacetime and its contents evolve from data on CD, the entire bubble interior is completely determined [at the classical level] by the properties of Δ and its contents. The "global point of view" criticised by advocates of complementarity denies this and, in effect, attempts to describe multiple copies of Δ simultaneously[6].

The connection with the *horizon* is as follows. The domain Δ is not spherical in shape. The horizon of a given observer must fit inside Δ, with no protrusion beyond

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[6]Note however that it has been suggested that the global and local points of view might themselves be complementary: see [35].
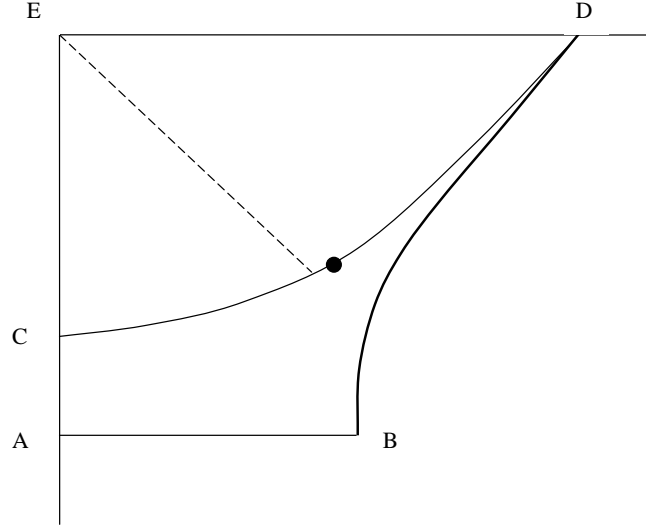
Figure 1: Thick-Walled Bubble Universe.

its boundary [since such a protrusion would entail a violation of complementarity if the horizon were to intersect horizons contained in other copies of the fundamental domain]. This gives an upper bound on the size of the horizon relative to Δ. However, it would be very unnatural if the horizon radius proved to be *much* smaller than the maximum. Ideally, we should be able to *prove* that Δ is *just barely* able to contain the horizon: in other words, we should find that Δ is positioned around the horizon in such a manner that the latter appears as one of the largest possible balls in Δ which do not protrude beyond the boundary. [In terms of the compactification, this is formulated by saying that the ball should be close to being as large as possible without self-intersection.] That is, horizon complementarity should be on or close to the brink of being violated. This is

in agreement with the principle put forward recently by Sekino and Susskind [36], who argue that [black hole] complementarity should "just barely escape inconsistency": there should be no "overkill". The situation is symbolized in Figure 1 by the placing of the dot *just outside* the cosmological horizon [represented by the diagonal dashed line] of the observer at the origin[7].

Now that we have a general picture of the kind of overall geometry for which we are aiming, we can try to be more specific. We begin with the observation that if we take care to set up spatial boundary conditions of physical fields in the fundamental domain Δ, in such a way that the description *excludes all reference to anything outside* Δ, then there will be physical consequences: we can expect the *Casimir effect* to arise [37][38]. The Casimir effect arising when negatively curved spaces are compactified in the manner proposed here has been studied extensively: see for example [39][40][41][42]. Typically, the Casimir effect has a local manifestation in terms of an energy density, which definitely couples to gravity [43], and which *violates the Null Energy Condition* or NEC. This property of the Casimir energy plays a crucial role here, since violations of the NEC often lead to instabilities of various kinds, and the need to avoid these instabilities yields useful constraints. The role of the Casimir effect in shaping the overall structure of the bubble spacetime is the theme of Section 3.

In order to be completely precise about this spacetime structure, we need to know the value of the Casimir coupling to gravity, specified here by a constant denoted by α. In principle, this can be worked out from the precise geometry of Δ and the detailed structure of the matter fields present at the relevant time. In practice this is an extremely difficult computation: even for compact *flat* spatial sections, which have a vastly simpler geometry, the computations involved are very intricate[8]. We shall however argue that *horizon complementarity may allow us to compute the value of this parameter* in a very simple way, as we shall shortly explain.

To proceed further, however, we need a more explicit description of the relative positions of the various objects portrayed in Figure 1. First, we need to specify the "size" of Δ, so that we can control the horizontal position of the heavy dot. Second, we need to know the [conformal] *time* at which the bubble spacetime emerges from the wall, relative to the [conformal] time corresponding to the horizontal line ED at the top of the diagram.

Regarding the size of Δ: we begin with the observation that hyperbolic space has *two* distinct length scales: one defined by the curvature, and the other by the volume of the smallest possible identical pieces into which it can be broken. The precise way in which H^3 should be broken up into minimal-volume "pieces" was settled recently in a major work due to Gabai et al. [51], who proved that the well-known *Weeks manifold*, a certain compactified version of three-dimensional hyperbolic space H^3, is the compactification with the smallest possible volume. The maximum possible radius of a ball in the Weeks manifold which does not self-intersect yields the second distinguished length scale for H^3.

As was mentioned earlier, there are good physical reasons to suppose that quantum-gravitational processes might favour spacetimes with compact negatively curved spatial

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[7]We do not place the dot *on* the horizon because the the distance to the boundary of Δ is direction-dependent. The situation shown represents a generic direction.

[8]There is an extensive literature on the use of the Casimir effect in cosmological models with compact flat spatial sections: see for example [44][45][46][47][48][49][50].

sections. This is in agreement with the intuition [32] that it should be "easier to create" small universes than large ones. Following this to its logical conclusion implies that, among the compact negatively curved three-dimensional spaces, the ones with least volume should be favoured. Furthermore, there is growing evidence [52] for Thurston's long-standing conjecture that there is a precise relationship between the combinatorial/topological *complexity* of a compact hyperbolic manifold and its volume, and it is reasonable to argue [53][54][55] that quantum-gravitational effects favour low complexity. Thus we are led to postulate that Δ is a fundamental domain for a Weeks manifold.

The answer to our second question, regarding the time represented by the surface CD, is less clear. However, there are two natural "landmarks" in the evolution of this spacetime, and it seems natural to assume that CD corresponds to one of them.

First, we shall see that there is a unique *minimal* spacelike hypersurface in the bubble spacetime; it owes its existence to the Casimir effect. It seems natural to assume that this is the surface along which a semi-classical description of the bubble interior first becomes appropriate; one might regard this as another application of the idea that "small" spatial sections are favoured. More physically, it has often been suggested [56][57] that the trace of the extrinsic curvature of spatial sections is the true measure of the passage of time in the earliest universe[9]. If this is true, then it is natural to suppose that the beginning of time should correspond to the vanishing of this "York time", and of course this corresponds to the minimal surface here. Finally, T-duality in string theory does suggest that the contracting part of any "bouncing" spacetime is probably a redundant description of some other, expanding part of the spacetime, so that zero York time is again favoured as the natural point for a semi-classical description to be appropriate.

There is however an alternative to this proposal. We shall take the matter content of the [earliest] bubble universe to consist of both an inflaton field and the Casimir energy. Because of the negative spatial curvature, the corresponding spacetimes contain two spacelike hypersurfaces along which the energy densities exactly cancel, so that the total energy of the universe vanishes at those times. One can certainly argue [59] that a semi-classical description should be established at such a time; furthermore, cutting off the spacetime at the second of these two surfaces has the advantage that the total energy of the bubble universe will then *never be negative*.

We shall consider both candidates for τ_e, the conformal time at which the semi-classical spacetime emerges from the bubble wall: that is, we shall examine the consequences of identifying the surface CD in Figure 1 with either the minimal surface or the [second] zero-total-energy surface.

These assumptions regarding Δ and τ_e are certainly the simplest one can make. The real test of their validity is however the predictions they can generate, and this is the main topic of the remainder of this work.

A direct application of horizon complementarity, as we have interpreted it here, imposes a strong constraint on the shape of the Penrose diagram for the "Inflation plus Casimir" spacetime discussed in section 3. This results in a *lower* bound for the Casimir parameter α. This is discussed in Section 4.

The Casimir effect entailed by horizon complementarity violates the Null Energy Con-

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[9]The fundamental importance of this quantity has also become apparent in recent deep analyses [58] of the nature of spacetime singularities.

dition [NEC]. In string theory, violations of the NEC often lead to a particular kind of instability, discovered by Seiberg and Witten [60]. By requiring that the system should not be unstable in this sense, and again using our assumptions regarding τ_e, we obtain an *upper* bound for α. This computation is the subject of Section 5.

Thus we see that horizon complementarity leads to both an upper and a lower bound on α. In fact, the allowed range for α is extremely narrow, particularly in the case in which $\tau_e = 0$. This is very remarkable, because although both constraints derive ultimately from horizon complementarity, one of them depends on purely geometric properties of the Weeks manifold and the other does not. Thus, in effect, we obtain a surprisingly precise prediction for the value of α, by means of two applications of horizon complementarity. This also confirms that the horizon just barely fits inside Δ, in agreement with our claim that the restriction to Δ accurately represents horizon complementarity, and with the Sekino-Susskind "no overkill" principle [36]. All this is discussed, with our conclusions, in Section 6.

3. The Role of the Casimir Effect

In this section, we shall introduce a simple explicit spacetime geometry, arising when the effects of Casimir energy are superimposed on the geometry of an inflating Coleman-De Luccia bubble universe. For the sake of clarity, let us recall that geometry.

The [simply connected version of] global de Sitter spacetime with [in the signature we use here] spacetime curvature $1/L^2$ is defined as the locus, in five-dimensional Minkowski spacetime [signature $(- + + + +)$], defined by the equation

$$- A^2 + W^2 + Z^2 + Y^2 + X^2 = L^2. \tag{1}$$

This locus has topology $\mathbb{R} \times S^3$, and it can be parametrized by *global* conformal coordinates $(\eta, \chi, \theta, \phi)$ defined by

$$
\begin{aligned}
A &= L \cot(\eta) \\
W &= L \operatorname{cosec}(\eta) \cos(\chi) \\
Z &= L \operatorname{cosec}(\eta) \sin(\chi) \cos(\theta) \\
Y &= L \operatorname{cosec}(\eta) \sin(\chi) \sin(\theta) \sin(\phi) \\
X &= L \operatorname{cosec}(\eta) \sin(\chi) \sin(\theta) \cos(\phi).
\end{aligned}
\tag{2}
$$

Here χ, θ, ϕ are the usual coordinates on the three-sphere, and η is angular conformal time, which takes its values in the interval $(0, \pi)$. The metric of Global de Sitter spacetime is then

$$g(\text{GdS}) = L^2 \operatorname{cosec}^2(\eta)[- d\eta^2 + d\chi^2 + \sin^2(\chi)\{d\theta^2 + \sin^2(\theta)d\phi^2\}]. \tag{3}$$

An obvious conformal transformation allows us to extend the range of η, so that it takes all values in the closed interval $[0, \pi]$. The Penrose diagram is clearly square [in the case of *simply connected* spatial sections], since χ also has this range.

Now notice that the defining formula (1) is invariant under an exchange, followed by a simultaneous complexification[10], of A and W; so this transformation cannot change

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[10]It is convenient to rotate A and W in opposite directions.

the local geometry. Therefore, if we perform the exchange and complexify both η [to complexify A and W] and χ [so as then to avoid complexifying X, Y, and Z], the resulting coordinates, defined by

$$
\begin{aligned}
\text{A} &= -\text{L}\operatorname{cosech}(\tau)\cosh(\sigma) \\
\text{W} &= \text{L}\coth(\tau) \\
\text{Z} &= \text{L}\operatorname{cosech}(\tau)\sinh(\sigma)\cos(\theta) \\
\text{Y} &= \text{L}\operatorname{cosech}(\tau)\sinh(\sigma)\sin(\theta)\sin(\phi) \\
\text{X} &= \text{L}\operatorname{cosech}(\tau)\sinh(\sigma)\sin(\theta)\cos(\phi),
\end{aligned}
\tag{4}
$$

are still coordinates on a spacetime locally identical to de Sitter spacetime. However, complexification will change the nature of the coordinates; the periodic coordinates are replaced by coordinates taking values in an infinite range. Thus the new conformal time coordinate τ ranges from zero to infinity, as does the coordinate σ which replaces χ. The effect of this is actually to *restrict* the domain of these new coordinates: they cannot cover the entire spacetime, because global de Sitter has compact spatial sections. Comparing the expressions for A in (2) and (4), we see that $\eta > \pi/2$ on the domain of these coordinates, and then a comparison of the two expressions for W shows that

$$
\eta > \frac{\pi}{2} + \chi.
\tag{5}
$$

We see that the new coordinates actually parametrise only one-eighth of the full Penrose diagram, the triangular top left-hand corner extending upwards from the point $\chi = 0$, $\eta = \pi/2$.

This is, however, all we need to describe the interior of the bubble portrayed in Figure 1, in the limit of an *infinitely thin* bubble wall [which corresponds to the null line in the conformal diagram above which the (τ, σ) coordinates are valid]. Thus it is reasonable to describe the region of de Sitter spacetime covered by these coordinates as "Bubble de Sitter". The metric in these coordinates is

$$
g(\text{BdS}) = \text{L}^2\operatorname{cosech}^2(\tau)[-\,d\tau^2 \,+\, d\sigma^2 \,+\, \sinh^2(\sigma)\{d\theta^2 \,+\, \sin^2(\theta)d\phi^2\}].
\tag{6}
$$

We see at once that this piece of de Sitter spacetime is foliated by spacelike hypersurfaces of constant *negative* curvature. One sees this also if one uses coordinates (t, r, θ, ϕ) based on proper time: the same metric is now

$$
g(\text{BdS}) = -\,dt^2 \,+\, \sinh^2(t/\text{L})\left[dr^2 \,+\, \text{L}^2\sinh^2(r/\text{L})\{d\theta^2 \,+\, \sin^2(\theta)\,d\phi^2\}\right];
\tag{7}
$$

here $r = \sigma\text{L}$. This replacement of the original spherical spatial sections by hyperbolic sections is well-known, and its importance has been particularly stressed by Freivogel et al. [4], as a prediction of the Landscape. Less appreciated, however, is that the complexification leading to this result *also renders conformal time infinite*. That is, τ runs from 0 [corresponding to $t = \infty$] to ∞ [as t tends to 0].

We have arrived at this conclusion regarding the range of conformal time inside the bubble because we have been studying the limiting situation in which the bubble wall is infinitely thin, as explained above. If we consider the more realistic situation portrayed in

Figure 1, with a thick bubble wall, then extremely large values of τ are cut off along the surface CD, corresponding to one or the other of the fixed "landmarks" described in the preceding section. Thus, the conformal diagram representing the semi-classical bubble interior will be rectangular, with the conformal time of emergence from the wall at the bottom. The corresponding value of τ will not be *infinitely* large. One should expect, however, that the diagram will be "very tall", in the sense that the range of τ will still be far larger than that of σ, if we restrict the latter's domain by compactifying the spatial sections. That would be unacceptable, since it is clear from Figure 1 that in fact the range of σ should [slightly] *exceed* the range of conformal time in the bubble.

On the other hand, we have also been assuming that the only energy density that is relevant here is the one contributed by the inflaton [in its potential-dominated state, so that it resembles a positive vacuum energy]. In reality, other forms of energy might be present, particularly in the region of the bubble wall — that is, precisely in the region of large τ. Can such additional matter have the effect of "shortening" the bubble's conformal diagram, as we need?

To answer this, we need the beautiful theorem due to Gao and Wald [61], which has a direct bearing on this question [see also [62]].

THEOREM [Gao-Wald]: Let M be a spacetime satisfying the Einstein equations and the following conditions:
[a] The Null Energy Condition [NEC] holds.
[b] M is globally hyperbolic and contains a compact Cauchy surface.
[c] M is null geodesically complete and satisfies the null generic condition.
Then there exist Cauchy surfaces S_1, S_2, with $S_2 \subset I^+[S_1]$, such that, for any $p \in I^+[S_2]$, one has $S_1 \subset I^-[p]$.

Here the null generic condition is the requirement that, along every null geodesic, there should exist a point where the tangent vector k^a and the curvature R_{abcd} satisfy $k_{[a} R_{b]cd[e} k_{f]} k^c k^d \neq 0$, and I^+, I^- denote respectively the chronological future [past] of an event or set of events; see [63], Chapter 8. The Null Energy Condition or NEC is the demand that the [full] stress-energy-momentum tensor should satisfy

$$T_{ab} n^a n^b \geq 0 \tag{8}$$

at all points in spacetime and for all null vectors n^a.

In simple language, the Gao-Wald theorem means that a sufficiently long-lived observer will, under the stated conditions, ultimately be able to "see" *an entire spatial slice of the spacetime.* An even simpler way of thinking about the theorem is as follows. Take simply connected global de Sitter spacetime, and note that the conclusion of this theorem is *not* true of it. This is because de Sitter spacetime is so "special" that it does not actually satisfy the null generic condition. In fact, because the conformal diagram is square, global de Sitter spacetime *just barely* escapes having a "fully visible" spatial section. Now suppose that one introduces into this spacetime a small amount of homogeneously distributed radiation. Then one can show explicitly [64] that the effect is to cause the conformal diagram to become "*taller*" [if we fix the width]. Of course, this affects all parts of the diagram, so even if we decide to cut off part of the spacetime at a finite time, the remaining part of the diagram is also stretched vertically when radiation is introduced.

The Gao-Wald theorem states that this happens quite generally: *generic matter that satisfies the NEC makes the conformal diagram of a spatially compact asymptotically de Sitter spacetime grow taller.*

We see that the mere inclusion of "other forms of energy" does not help us; in fact, "normal" matter just makes the situation worse. The simplest way to circumvent the Gao-Wald theorem is to *violate the NEC*. We need to find a natural way of doing this in the immediate vicinity of the bubble wall, without interfering with the subsequent Inflation. We shall now argue that our proposal — to impose a formal compactification of the spatial sections inside the bubble — automatically supplies the needed mechanism.

Horizon complementarity requires us to give a complete description of the physics inside the [inflationary] horizon, using *only* those fluctuations which are completely contained in a fundamental domain Δ, of the kind discussed in the preceding section. Thus, complementarity requires us to impose boundary conditions of precisely the kind which lead to the *Casimir effect* [37][38]. This effect can be represented in a formal way by including a negative energy component, which is however non-negligible only when the universe is extremely small. As is well known, the Casimir effect violates the NEC, and so it provides us with precisely what we want: a natural, indeed inevitable, way of evading the Gao-Wald theorem.

Let us set up a simple Friedmann model of this system; in doing so, we are as usual ignoring the back-reaction of the inhomogeneities in the Casimir energy distribution. [In the case of the Weeks manifold with which we are concerned here, the relative inhomogeneities of the Casimir energy tend in any case to be very mild; see [39].] We shall work with the usual FRW spacetime geometry, with scale factor a(t), where t is proper time, and with negatively curved spatial sections on which we continue to use the coordinates (r, θ, ϕ). [The fact that these sections have been compactified is not apparent in the form of the local metric; it is reflected only in the range of r, assuming that we take this coordinate to be single-valued. See below.] Various kinds of physical fields and compactification schemes contribute to the Casimir energy in various ways and with different signs; we shall assume that the total is *negative* and depends on the inverse fourth power of the scale factor — see for example [47]. This is to be combined with a positive vacuum energy density $\rho_{\text{inflaton}} = +3/8\pi L^2$, representing the inflaton in its potential-dominated state; here L is the typical inflationary length scale [that is, the spacetime curvature at the end of Inflation is $1/L^2$], and we are using Planck units.

It will be convenient to represent the Casimir coupling to gravity in the following way. Let us declare that the ratio of the magnitude of the Casimir energy density to that of the inflaton is given by a dimensionless positive constant γ at the time when the scale factor is equal to unity [assuming for the moment that this actually occurs]. Thus, the Casimir energy density ρ_{casimir} is given by $-3\gamma/8\pi L^2 a^4$. The Friedmann equation is

$$L^2 \dot{a}^2 \;=\; \frac{8\pi}{3} L^2 a^2 \Big[\rho_{\text{inflaton}} \,+\, \rho_{\text{casimir}}\Big] \,+\, 1 \;=\; \frac{8\pi}{3} L^2 a^2 \Big[\frac{3}{8\pi L^2} \,-\, \frac{3\gamma}{8\pi L^2 \, a^4}\Big] \,+\, 1. \quad (9)$$

The solution for the "Bubble de Sitter plus Casimir" metric [with the constant of integration absorbed into the time coordinate] is

$$g(\text{BdS}\alpha) = -\,dt^2 + \Big[\alpha^2 + (1\,+\,2\alpha^2)\sinh^2(t/L)\Big]\Big[dr^2 + L^2\sinh^2(r/L)\{d\theta^2 + \sin^2(\theta)\,d\phi^2\}\Big], (10)$$

where

$$\alpha^2 = \sqrt{\frac{1}{4} + \gamma} - \frac{1}{2}. \tag{11}$$

Note that α is the smallest possible value of the scale factor. This gives the geometric meaning of α: if for example Δ is a fundamental domain for the Weeks manifold, then [see the next section] the minimal volume of a spatial section in this spacetime is approximately $0.9427\alpha^3 L^3$, where L is the inflationary length scale. We can now also clarify the physical meaning of α, as follows. Recall that γ was defined as a "coupling" which measures the relative Casimir and inflaton energy densities at the time when the scale factor equals unity. However, $a(t) = 1$ has no particular significance in this problem [and indeed it may not occur at any time], so the physical meaning of γ is obscure. We now see, however, that the maximal magnitude of the Casimir energy density [which is clearly attained at a unique time, $t = 0$] is $|\rho^{\text{max}}_{\text{casimir}}| = 3\gamma/8\pi L^2 \alpha^4$, and in fact one has

$$|\rho^{\text{max}}_{\text{casimir}}| = \frac{3}{8\pi L^2}(1 + \alpha^{-2}). \tag{12}$$

Thus α can be thought of as a number which parametrises the maximum intensity of the Casimir effect in this problem. This provides the physical meaning of this parameter. Henceforth we shall refer to α as the "Casimir coupling", since it is both geometrically and physically more meaningful than γ. [In fact, $\gamma = \alpha^2 + \alpha^4$, so we can regard γ as this function of α.]

Note that the maximal value of the magnitude of the Casimir energy density is, from (12), always somewhat larger than the inflaton energy density [provided that the hypersurface $t = 0$ is indeed retained as part of the spacetime geometry — see below]. Thus, the *total* energy density is negative near to $t = 0$. As the Casimir energy is diluted, the total energy density passes through zero, when the scale factor is equal to $\gamma^{1/4}$ [though the total *pressure* is strictly negative even at that time] and is thereafter positive. The formal "total equation of state parameter" [the ratio of total pressure to total energy density] is given by

$$w_{\text{tot}} = \frac{-(1 + \gamma/3a^4)}{(1 - \gamma/a^4)}, \tag{13}$$

where $a(t)$ is the scale factor. Notice that, immediately after the time at which the total energy density vanishes, w takes on arbitrarily large negative values. [It then rapidly approaches -1.] This observation is of some interest in the light of the recent work of Steinhardt and Wesley [65], who argue that Inflation can only arise in high-dimensional theories if w does assume such values; that is, if the system is "deep in the NEC-violating regime". We see that Inflation *automatically satisfies this condition* if the Casimir effect is present[11].

It will be useful for us to write our metric in terms of [dimensionless] conformal time, τ, which is given by

$$\tau = \frac{1}{\alpha}\int_0^{t/L} \frac{dt/L}{\sqrt{1 + [2 + (1/\alpha^2)]\sinh^2(t/L)}} = \frac{-i}{\alpha} F\left(\frac{it}{L}, \sqrt{2 + (1/\alpha^2)}\right). \tag{14}$$

---

[11]Steinhardt and Wesley show that many other conditions need to be satisfied; we are *not* claiming that our observation discussed here settles all of their concerns. [Nor does the fact that they assume flat spatial sections mean that similar results cannot be proved for bubble interiors; that remains to be seen.]

Here $F(\phi, k)$ is the incomplete elliptic integral of the first kind [66], with Jacobi amplitude ϕ and elliptic modulus k; in this case it has been evaluated along the imaginary axis.

Inverting the elliptic integral we can express t in terms of the amplitude:

$$it/L = am\left(i\alpha\tau, \sqrt{2 + (1/\alpha^2)}\right). \tag{15}$$

Taking the sine of both sides we find that

$$i\sinh(t/L) = sn\left(i\alpha\tau, \sqrt{2 + (1/\alpha^2)}\right), \tag{16}$$

where $sn(u, k)$ is one of the classical Jacobi elliptic functions. Using the formulae for complex arguments of elliptic functions given on page 592 of [66], one can express the right side as a function of a real variable; substituting the result for $\sinh(t/L)$ in equation (10) we obtain finally [with $\sigma = r/L$]

$$g(BdS\alpha) = L^2\left[\alpha^2 + (1 + 2\alpha^2)\frac{sn^2\left(\alpha\tau, i\sqrt{1 + (1/\alpha^2)}\right)}{cn^2\left(\alpha\tau, i\sqrt{1 + (1/\alpha^2)}\right)}\right]$$
$$\times \left[-d\tau^2 + d\sigma^2 + \sinh^2(\sigma)\{d\theta^2 + \sin^2(\theta)d\phi^2\}\right]. \tag{17}$$

Here $cn(u, k)$ is another of the Jacobi elliptic functions. This metric is to be compared with the Bubble de Sitter metric given in equation (6); the metric here is conformally the same as that metric along future infinity; it is asymptotically de Sitter. [In fact, the spacetime geometry here is significantly different from that of Bubble de Sitter spacetime *only* for a short time after the emergence from the bubble wall, since the Casimir energy is diluted away very rapidly with the expansion.]

The geometry here differs from that of Bubble de Sitter in an important way, however: the extent of conformal time is *not* infinite. Its extent is instead given by setting the elliptic function $cn\left(\alpha\tau, i\sqrt{1 + (1/\alpha^2)}\right)$ equal to zero. The zeros of this function are given [[66], page 590] by

$$cn(K(k), k) = 0, \tag{18}$$

where $K(k)$ is the *complete elliptic function of the first kind*. Thus τ has a formal range between

$$\pm\tau_\infty(\alpha) = \pm\frac{1}{\alpha}K\left(i\sqrt{1 + (1/\alpha^2)}\right). \tag{19}$$

Using the relevant formula from page 593 of [66] one can express this in terms of real variables:

$$\pm\tau_\infty(\alpha) = \pm\frac{1}{\sqrt{1 + 2\alpha^2}}K\left(\sqrt{\frac{1 + \alpha^2}{1 + 2\alpha^2}}\right). \tag{20}$$

However, the negative value here corresponds to proper time t tending to $-\infty$, which is not correct; it should instead be replaced by the conformal time τ_e at which the bubble universe *emerges* from the bubble wall. Our hypothesis, explained in the preceding section, is that τ_e either vanishes or corresponds to the moment when the total energy of the universe is zero. We shall return to that later; for the moment, let us note that since the function $K(k)$ diverges as k tends to unity [it increases monotonically from a value of $\pi/2$ at k = 0, and in particular is finite at $1/\sqrt{2}$ — see [66], page 592], $\tau_\infty(\alpha)$ can be arbitrarily large
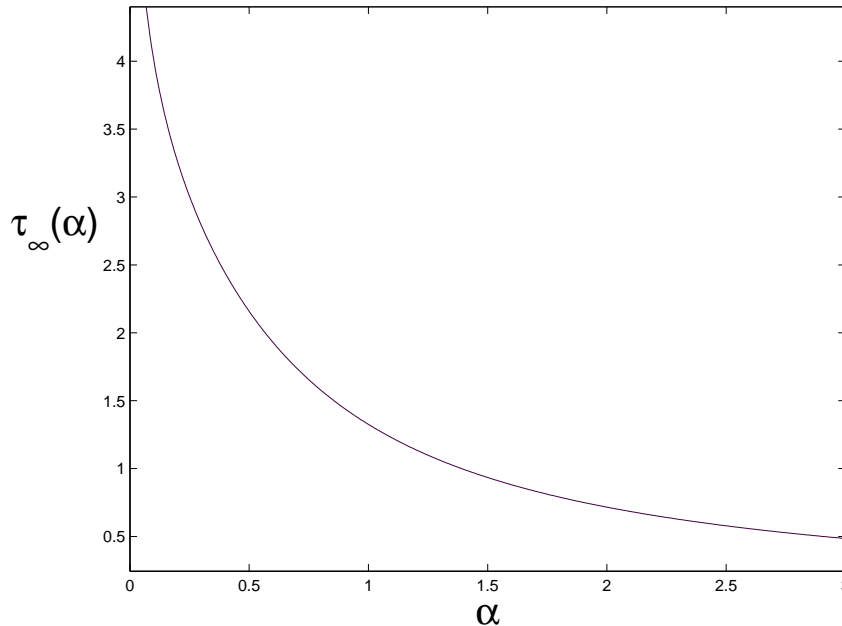
13

Figure 2: Conformal Time to Infinity as a Function of α.

if α is very small, or arbitrarily small if α is large; see Figure 2. Thus, we see that the shape of any conformal diagram for this spacetime depends strongly on α; and that, by adjusting α, we can avoid having a conformal diagram which is too tall.

This is a very satisfactory conclusion. We saw that the original bubble de Sitter spacetime has a conformal diagram which is extremely "tall", and that the presence of matter satisfying the NEC only made the situation worse. However, our proposed interpretation of horizon complementarity automatically leads us to include, in addition to the inflaton, another form of energy — Casimir energy — which allows us to evade the conclusions of the Gao-Wald theorem and to restrict the range of conformal time in just the way demanded by Figure 1. In the next section we will show how to be more precise about this.

We see, then, that horizon complementarity leads to a rather specific description of the spacetime geometry of the earliest part of the history of a bubble universe. All that remains is to determine the constant α. We shall now argue that horizon complementarity also constrains this parameter very strongly.

4. The Geometry of the Weeks Manifold: First Constraint on α

It was shown long ago by Thurston [67] that there exists a decomposition of hyperbolic three-space H^3 into identical pieces of minimal volume — that is, that there is a minimal-volume compactification. It was long conjectured, and finally proved by Gabai et al. [51], that this distinguished decomposition corresponds to the *Weeks manifold*[12], $\mathbf{W}$. For this manifold, the fundamental domain can be represented as a certain hyperbolic polyhedron with 18 faces. The volume is $\approx 0.9427 \times \lambda^3$, where λ is the curvature radius.

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[12]A good description of the Weeks manifold, with illustrations, may be found in [68]; see also [69].

Our hypothesis is that this is the hyperbolic compactification which defines the domain Δ.

Recall that we propose to relate the fundamental domain to the cosmic horizon [at the time of emergence from the bubble wall] by identifying the horizon as a sphere which is completely enclosed by the domain. The systematic study of such spheres can be briefly explained as follows.

In any compact manifold M, the *injectivity radius* I(M, p) at a point p in M is defined as the maximal radius of a sphere centred at p which does not self-intersect. [That is, the maximal radius such that the exponential map is injective.] For a sphere or a torus, this quantity is actually independent of the point p, but *this is not so* for compact hyperbolic manifolds, for which the boundary of a fundamental domain is much more irregular. That is, the injectivity radius is a function of position on a compact hyperbolic manifold.

The range of sizes of spheres which can be contained in a compact hyperbolic space can be surveyed as follows. For each such space one can define an *injectivity distribution*, a function introduced by Weeks [70] and defined as follows. Let dV/V be the fraction of the volume of M containing points p with I(M, p) [measured in units of λ] lying between the values x and x + dx. Then the injectivity distribution is the function on the real line defined by

$$ID(M; x) \;=\; (dV/V)/dx. \tag{21}$$

That is, ID(M; x) measures the rate at which the fractional volume containing points with a given injectivity radius changes with increasing injectivity radius; integrating it between selected values of x gives the fraction of the volume of M containing points with injectivity radii between those values. The curve representing ID(M; x) [which need not be a continuous function] intersects the x axis at two points. The smaller of these two values signals the radius at which it becomes possible for a sphere to self-intersect; the larger signals the radius beyond which this must happen. This latter quantity is sometimes called the *inradius* of M, which we denote by $\sigma(M)$. [Here σ refers to the dimensionless radial coordinate used in equation (17).]

The functions ID(M; x) are given in approximate form for ten low-volume hyperbolic spaces in [70]. In particular, for the Weeks manifold **W** ["Manifold 1" in [70]], ID(**W**; x) is a function which has support on an interval extending roughly from 0.292 to 0.519 $\approx \sigma(\mathbf{W})$. Thus, a horizon of conformal radius less than 0.292 can be located anywhere in H^3/Γ without danger of intersecting itself; but a horizon of conformal radius larger than 0.519 would have to do so, no matter where it might be located. [Note that this last quantity, the inradius, varies between roughly 0.5 and 0.6 for the ten low-volume manifolds examined in [70]; however it can be substantially larger than this for other well-known compact hyperbolic manifolds; it is approximately 0.996 for the Seifert-Weber space, the most easily visualised compact hyperbolic manifold [26].]

From the above discussion of the injectivity distribution, one sees that the spacetime with local metric given in (17) and spatial sections of the form H^3/Γ does not have a Penrose diagram in the conventional sense, because the fundamental domain Δ is neither [globally] rotationally symmetric nor [globally] homogeneous. That is, if we take the metric in (17) to be defined on H^3/Γ, then the range of the conformal coordinate σ, while always finite if we demand that it be single-valued, depends on position and orientation. Therefore, one has a collection of diagrams, one for location in Δ and one for each direc-

tion; each is rectangular, with the same height [for given assumptions regarding the range of conformal time] but with varying widths. This situation is represented by the "Penrose diagram" in Figure 3, where the dotted rectangles represent possible diagrams for various directions and positions. The axes here correspond to the conformal coordinates τ and σ in (17); the diagram represents the spacetime geometry when the overall conformal factor has been removed.
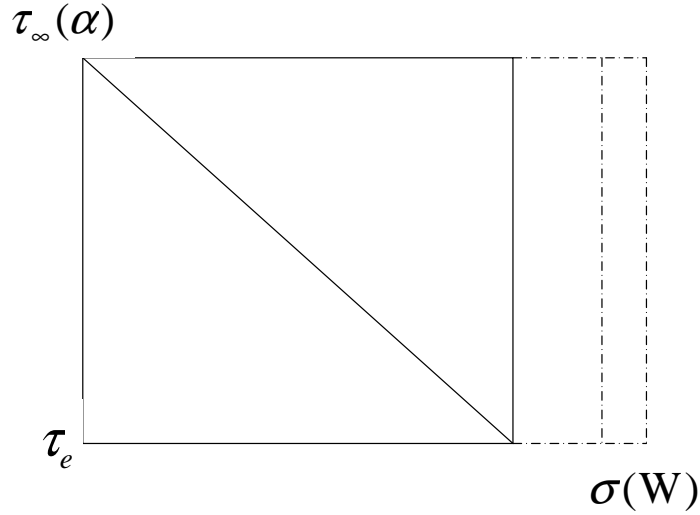
$$\tau_\infty(\alpha)$$

$$\tau_e$$

$$\sigma(\mathbf{W})$$

Figure 3: "Penrose Diagram" for a Spatially Compactified Bubble.

The coordinate τ ranges from its value τ_e at the time of emergence from the bubble wall up to $\tau_\infty(\alpha)$. The coordinate σ ranges from zero up to a certain maximum; the extreme case of interest here corresponds to a sphere located at a point where the injectivity radius is as large as possible, taking the value $\sigma(\mathbf{W})$. The horizon is, as usual, the diagonal line

in the figure. Since we insist that the horizon must never, at any location in Δ, extend beyond the latter's boundary, we have a simple inequality:

$$\tau_\infty(\alpha) \; - \; \tau_e \; < \; \sigma(\mathbf{W}), \tag{22}$$

though, as we have explained, we would certainly prefer to replace the inequality by an [approximate] equality.

We have two candidate assumptions for τ_e. The first is that it should be taken to be zero: we accept exactly the expanding half of the original spacetime. In that case, we have

$$\tau_\infty(\alpha) \; < \; \sigma(\mathbf{W}) \; \approx \; 0.519. \qquad [\tau_e = 0] \tag{23}$$

A numerical investigation reveals that equation (20) now implies

$$\alpha \; > \; 2.80, \qquad [\tau_e = 0] \tag{24}$$

approximately. This can be checked by inspecting the graph in Figure 2. Note that because the graph is quite flat for values of α beyond about 2, the replacement of the Weeks manifold by a manifold [if one existed] with even a slightly smaller inradius would strengthen this constraint significantly; on the other hand, its replacement by [say] the Seifert-Weber space [inradius approximately 0.996] would weaken the constraint on α somewhat less dramatically.

The alternative hypothesis is that τ_e corresponds to the conformal time $\tau_{\rho=0}$ when the total energy density of the universe is zero. This occurs when the scale factor increases from its minimal value [given by $a(0) = \alpha$] to

$$a(t_{\rho=0}) \; = \; [\alpha^2 \; + \; \alpha^4]^{1/4}, \tag{25}$$

where $t_{\rho=0}$ is the proper time corresponding to $\tau_{\rho=0}$. Using the scale factor as the variable, we can show after a short calculation that (23) must be replaced by

$$\frac{1}{\sqrt{1 \; + \; 2\alpha^2}} \, \mathrm{K}\left(\sqrt{\frac{1 \; + \; \alpha^2}{1 \; + \; 2\alpha^2}}\right) \; - \; \int_\alpha^{[\alpha^2 \, + \, \alpha^4]^{1/4}} \frac{\mathrm{d}a}{\sqrt{a^4 \; + \; a^2 \; - \; \alpha^2 \; - \; \alpha^4}} \; < \; \sigma(\mathbf{W}), \; \; [\tau_e = \tau_{\rho=0}] \tag{26}$$

where we remind the reader that $\mathrm{K}(k)$ is a certain elliptic function.

Although this is far from obvious, a rather more involved numerical investigation shows that the complicated function of α on the left side of this inequality is still a decreasing function; hence (26) still yields a lower bound on α. We find that the constraint on α is now somewhat weaker:

$$\alpha \; > \; 2.42. \qquad [\tau_e = \tau_{\rho=0}] \tag{27}$$

Of course, both (24) and (27) still allow for very large values of α. That would correspond to horizon radii which are very much smaller than they need to be in order to fit inside the domain Δ. Since the decomposition of the spatial sections into copies of Δ is supposed to be a formal description of horizon complementarity, that would be a disappointing conclusion, and furthermore it would conflict with Sekino and Susskind's "no overkill" principle [36]; it would be far preferable to find that the horizon just barely fits inside Δ. We shall now argue that this is, in fact, exactly what happens.

5. Non-Perturbative Stability: Second Constraint on α

As we have seen, it is essential for our purposes that the Casimir effect violates the Null Energy Condition. It is well known, however, that such violations often lead to serious instabilities. We now consider this.

The status of the NEC has been much debated of late, from various points of view [see for example [71][72][73] and references therein]. NEC violation may or may not be acceptable in cosmology — the recent work of Steinhardt and Wesley [65] suggests that it may simply be *unavoidable* if Inflation is embedded in a higher-dimensional theory — but it is certainly the case that there are many circumstances in which it leads to major problems [due to ghosts and gradient energies of the wrong sign [74][75]]. Exceptions do arise, however, in strictly quantum, non-local systems [76].

One such exception occurs around a black hole which is not in equilibrium with its own Hawking radiation [77]; in that case, the NEC violation is associated with the "quantum defocussing" which allows the event horizon to contract. The second main example of physically acceptable NEC violation is provided precisely by the Casimir effect we have been discussing. Arkani-Hamed et al. argue [76] that this intrinsically quantum effect is acceptable because one cannot use it to construct "non-gravitating clocks and rods."

However, while Casimir energy may be acceptable in most cases, this is not to say that it is always completely innocuous. In particular, since the work of Seiberg and Witten [60], it has been clear that branes and other extended objects can lead to forms of instability which are completely obscure in any perturbative approach. [More complex kinds of non-perturbative instability are also known [78].] It is not obvious that Casimir energy is always harmless in the non-perturbative context.

Seiberg-Witten instability refers to the uncontrolled nucleation of branes in certain modified versions of anti-de Sitter spacetime; it depends on a delicate interplay between the growth of volumes and surface areas in asymptotically hyperbolic Euclidean geometries[13]. Unfortunately, there is no known general criterion which might allow us to decide, without a detailed calculation, whether a given system is free of Seiberg-Witten instabilities; normally one has to evaluate the relevant action explicitly, and show for example that it grows monotonically away from a given non-negative value. Explicit examples of systems which are unstable in this sense were discussed by Maldacena and Maoz [79]; see also [80][64]. The method of direct examination of the action was also used recently to rule out AdS black holes with certain exotic event horizon topologies [81] and to verify the non-perturbative stability of certain solutions of M-theory [82].

While no completely general criterion for the occurrence of Seiberg-Witten instability is known, many concrete examples show that NEC violation does have a *tendency* to induce it [83]. Thus, we must take care to ensure that this does not occur in our case. To put it another way: by *requiring* that it should not occur, we may obtain a useful constraint.

In four spacetime dimensions the brane action has, for BPS branes, the general form

$$S = \Theta\left(A - \frac{3}{L}V\right), \tag{28}$$

---

[13]A Riemannian manifold is said to be *asymptotically hyperbolic* if it has a well-defined conformal boundary. The methods introduced by Seiberg and Witten apply to any spacetime with an asymptotically hyperbolic Euclidean version.

where Θ is the tension, A is the brane area, V the volume enclosed, and L is the background asymptotic curvature radius. It is clear that this object can be *everywhere* non-negative, as it must be if Seiberg-Witten instability is to be avoided, only in a very particular kind of geometry: one where the area grows extremely rapidly with radius.

Let us see how this works in the simplest possible asymptotically hyperbolic geometry, namely four-dimensional hyperbolic space H^4 itself. Here it is easy to show that the action is positive at *large* distances, but the situation is less clear at small distances from the origin[14]. The metric here, using the global foliation of H^4 by three-spheres, is

$$g(H^4) \;=\; dt^2 \;+\; L^2 \sinh^2(t/L) \left[d\chi^2 \;+\; \sin^2(\chi)\{d\theta^2 \;+\; \sin^2(\theta)\, d\phi^2\} \right], \qquad (29)$$

and the brane action is

$$S[H^4](\Theta,\, L\,;\, t) \;=\; 2\pi^2 \,\Theta\, L^3 \left[\sinh^3(t/L) \;-\; \frac{1}{4}\cosh(3t/L) \;+\; \frac{9}{4}\cosh(t/L) \;-\; 2 \right]. \quad (30)$$

Note that the negative second term here is actually *larger* in magnitude than the first term for small values of t, underlining the fact that the positivity of the action is somewhat precarious even here, in the case of undisturbed "pure" hyperbolic space. Nevertheless, one can verify that [because of the presence of the third term, which is negligible at large distances] this function, which obviously vanishes at $t = 0$, is monotonically increasing, and hence is everywhere non-negative; see Figure 4. [In this and in the subsequent diagrams, the t-axis has units given by L; the units on the vertical axis are given by $2\pi^2\,\Theta\, L^3$.]
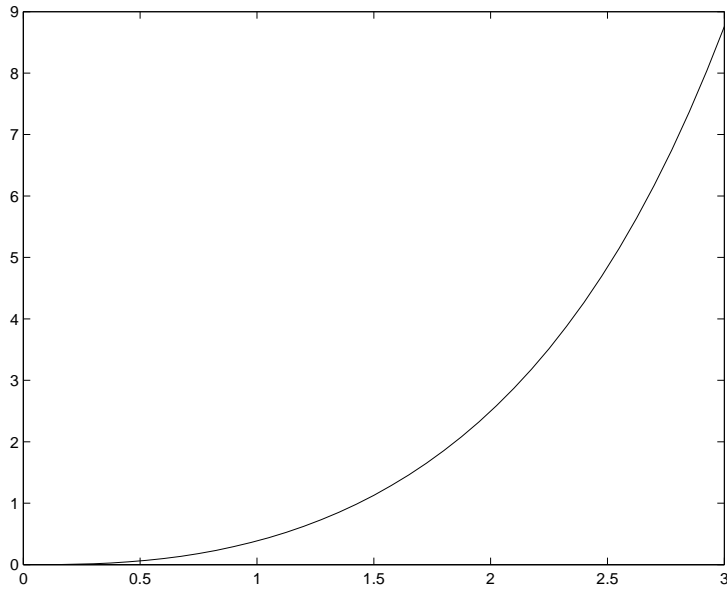

Figure 4: Brane Action for Pure H^4.

Clearly, however, one would not need to modify the geometry of hyperbolic space very drastically in order to cause the brane action to become negative near to the origin.

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[14]Seiberg and Witten gave a simple criterion [positivity of the scalar curvature of the boundary] for the brane action to be positive at *large* distances.

The Seiberg-Witten brane action is in fact a much more subtle object than its simple appearance might suggest.

One modification which certainly has a clear potential to lead to trouble here is to introduce a "wormhole-like" structure into H^4, that is, some kind of structure such that a large volume is contained in a small area; for this will allow the volume term in (28) to dominate the area term in the vicinity of the "wormhole-like" object. In general terms, there are strong suggestions from several directions [84][85][86][87] that such objects are in fact unacceptable in quantum gravity. The requirement of Seiberg-Witten stability allows us to be more precise in the case of interest to us.

The point is that the Euclidean geometry corresponding to our metric (10) is indeed "wormhole-like": to be precise, it corresponds to *part of* a Euclidean wormhole. [This is of course not surprising, in view of the familiar association of wormholes with NEC violation.] To see this, note that the metric given in (29) is precisely the asymptotically hyperbolic Euclidean version of the Bubble de Sitter metric (7): one obtains (29) from (7) simply by complexifying t and L and re-labelling r. Performing this same complexification on (10), we obtain the asymptotically hyperbolic Euclidean space with metric

$$
g(\text{AHBdS}\alpha) = dt^2 + L^2 \left[\alpha^2 + (1 + 2\alpha^2) \sinh^2(t/L) \right]
$$
$$
\times \left[d\chi^2 + \sin^2(\chi)\{d\theta^2 + \sin^2(\theta)\, d\phi^2\} \right]. \quad (31)
$$

If we allow t to run from $-\infty$ to $+\infty$, then this is like two copies of H^4 joined smoothly near the origin by a wormhole. But clearly the part of the space corresponding to *negative* values of t will lead to problems here: if that part of the wormhole is retained, then we run the risk that some branes at positive t will have small areas and large volumes, causing the brane action to become negative. We see that non-perturbative stability actually *forces* us to truncate the manifold at some value $t = t_e$ of the proper time, corresponding to the emergence of a semi-classical bubble spacetime from the bubble wall. In view of our discussion in the preceding section, this is very satisfactory; we now have a method of assessing the physical consequences of truncating the spacetime in either of the two ways discussed in the previous section.

For the metric given in equation (31), the brane action for $t \geq t_e$ is

$$
S[\text{AHBdS}\alpha](t_e, \Theta, L; t) = 2\pi^2 \Theta L^3 \left[\left(\alpha^2 + (1 + 2\alpha^2) \sinh^2(t/L) \right)^{3/2} \right.
$$
$$
\left. - \frac{3}{L} \int_{t_e}^{t} \left(\alpha^2 + (1 + 2\alpha^2) \sinh^2(u/L) \right)^{3/2} du \right], (32)
$$

where Θ is the tension, as in equation (28). As the metric here is asymptotically indistinguishable from that of the pure hyperbolic space discussed above, this function is certainly positive at large t. The problem is to understand what happens at small values of t, where the NEC violation is most intense.

The derivative of the action with respect to t can be expressed, after a straightforward calculation, as

$$
\frac{dS[\text{AHBdS}\alpha](t_e, \Theta, L; t)}{dt} = 3\pi^2 \Theta L^2 \left[1 - (1 + 2\alpha^2) e^{-2t/L} \right]
$$
$$
\times \left[\alpha^2 + (1 + 2\alpha^2) \sinh^2(t/L) \right]^{1/2}. \quad (33)
$$

Let us consider our first proposal for the truncation, along the hypersurface of zero extrinsic curvature $t_e = \tau_e = 0$; we return to the alternative, a truncation along a surface of zero total energy density, below. We begin by stressing that the initial value of the action is not zero for $\alpha > 0$: it is equal to the positive value $2\pi^2 \Theta L^3 \alpha^3$. Furthermore, one can show that the second derivative is positive everywhere. On the other hand, we see at once that the slope of the graph of the action is *negative* at $t = 0$; it is equal to $-6\pi^2 \Theta L^2 \alpha^3$. The action function is *not* monotonically increasing, as it is in the case of pure hyperbolic space; as we feared, there is a real possibility that the action could become negative. It can be shown that this initial decrease of the brane action is due to the fact that *the Casimir effect violates the NEC.* We see that NEC violation tends to induce Seiberg-Witten instability, as claimed.

The graph of the action reaches a unique minimum [as can be seen from (33)] at a positive value of t, namely $t = (\ln(\sqrt{1 + 2\alpha^2}))L$. The system will be stable in the Seiberg-Witten sense provided that the action is non-negative at this point. That is, if we define a number Ξ_α, depending only on α, by

$$\Xi_\alpha = \frac{S[AHBdS\alpha](t_e = 0, \, \Theta, \, L \, ; \, (\ln(\sqrt{1 + 2\alpha^2}))L)}{2\pi^2 \Theta L^3}, \qquad (34)$$

then the system is non-perturbatively stable if and only if $\Xi_\alpha \geq 0$.

In fact, a numerical investigation shows that the action *does* become negative if α is sufficiently large, showing that Seiberg-Witten instability is a possibility here even in the absence of half of the wormhole. For example, if we take $\alpha = 5$, then it is clear from Figure 5 that the system will be unstable in the Seiberg-Witten sense. If we think of Ξ_α
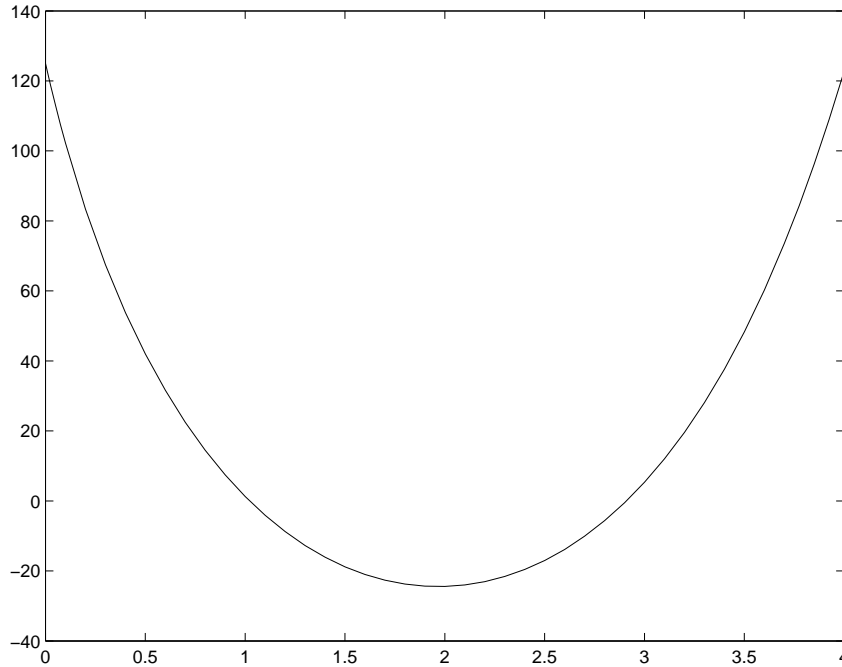

Figure 5: Brane Action, $\alpha = 5$, $\tau_e = 0$.

as a function of α, then we see that this function is already negative at $\alpha = 5$, and in fact it becomes steadily more negative as α increases beyond 5. We know that Ξ_α is zero

at $\alpha = 0$ [Figure 4], and it would be perfectly reasonable to expect that it is negative for *all* positive α; this would mean that NEC violation leads to instability in all cases, which would not be entirely unexpected. Remarkably, however, numerical experiments show
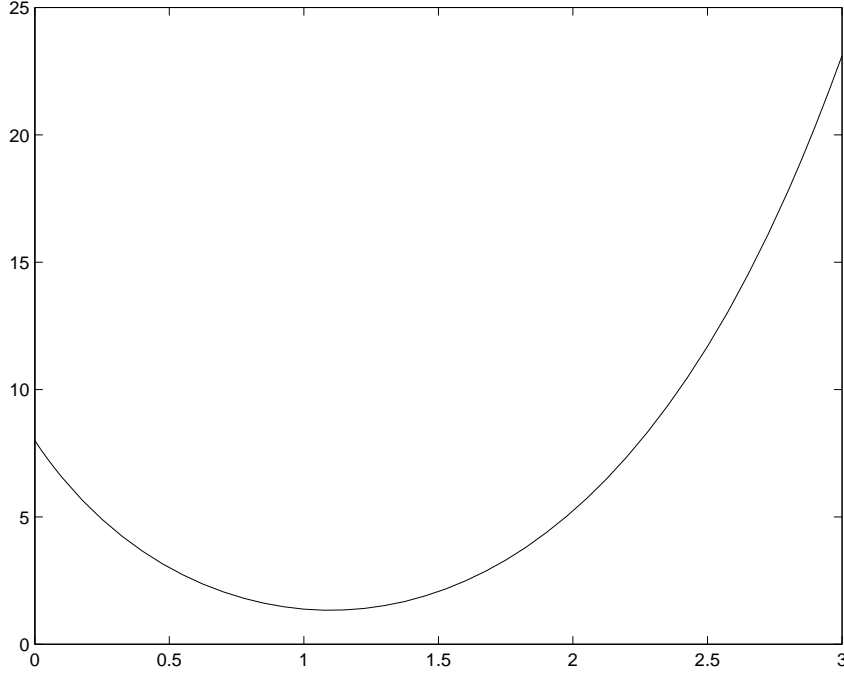

Figure 6: Brane Action, $\alpha = 2$, $\tau_e = 0$.

that this is *not* the case: as a function of α, Ξ_α is actually positive for a brief interval near to $\alpha = 0$. For example, it is positive at $\alpha = 2$: see Figure 6. Further experimentation shows that Ξ_α is approximately zero at around $\alpha = 2.88$, as shown in Figure 7. Thus, despite the NEC violation associated with the Casimir effect, the system is actually non-perturbatively *stable* for values of α between zero and about 2.88, but not beyond that value; so we have

$$\alpha \;<\; 2.88. \qquad\qquad [\tau_e = 0] \qquad\qquad (35)$$

Let us consider our alternative proposal for τ_e, defined now as the conformal time at which the total energy density vanishes. This is a more complex situation computationally, because now τ_e and t_e are themselves functions of α, as one can see from equation (25). Thus the action given in equation (32) acquires a still more complicated dependence on α. We find numerically [again, the use of the scale factor as the variable is helpful] that requiring the action to be non-negative in this case still imposes an upper bound on α, albeit a slightly weaker one: (35) is replaced by

$$\alpha \;<\; 2.91. \qquad\qquad [\tau_e = \tau_{\rho=0}] \qquad\qquad (36)$$

We see that the zero-density definition of τ_e imposes weaker conditions on α, both from above and from below, than the zero-extrinsic-curvature definition; though the effect on the upper bound is much smaller than that on the lower bound.
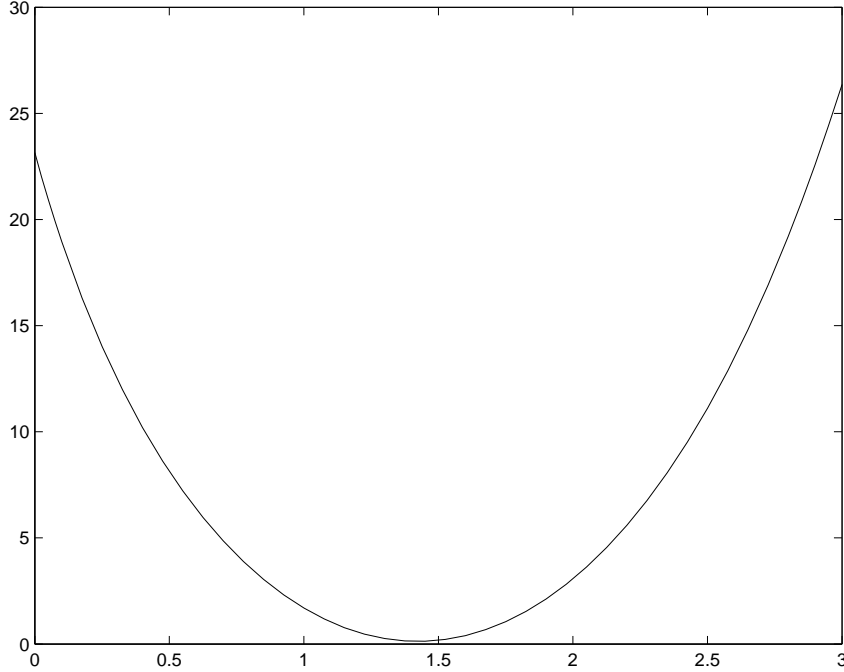
Figure 7: Brane Action, $\alpha = 2.88$, $\tau_{\mathrm{e}} = 0$.

6. Conclusion: Using Horizon Complementarity to Predict α

We can summarize as follows. We have argued that horizon complementarity should be implemented in the case of bubble universes by means of the scheme represented by Figure 1. It turns out that this requires the presence of some other form of energy, apart from that of the inflaton, in the earliest history of the bubble world; but fortunately the compactification itself supplies a natural way of ensuring this, through the Casimir effect. Requiring the conformal diagram to take a shape consistent with Figure 1 imposes a *lower* bound on the gravitational Casimir coupling. But violating the NEC forces us to impose an *upper* bound on the coupling if non-perturbative instability is to be avoided. With our two natural candidates for the time of emergence of the bubble universe from the bubble wall, we obtain combined constraints of the form [see the inequalities (24)(35)(27)(36)]

$$2.80 \; < \; \alpha \; < \; 2.88, \qquad\qquad [\tau_{\mathrm{e}} = 0] \qquad\qquad (37)$$

$$2.42 \; < \; \alpha \; < \; 2.91. \qquad\qquad [\tau_{\mathrm{e}} = \tau_{\rho = 0}] \qquad\qquad (38)$$

Inserting a value of α in one of these ranges into equation (10), we have a completely definite form for the metric of the bubble spacetime during its earliest history. We see that while the compactification may only be enforced beyond the horizon, it does strongly constrain the local metric: in fact, the metric is nearly fixed.

The constraints we have obtained on α are remarkably tight, astonishingly so in the $\tau_{\mathrm{e}} = 0$ case. However, the reader is entitled to object that the precision of the resulting prediction for α, while very striking, is an illusion: clearly we have made approximations in order to arrive at these results, and it is not obvious how these can be quantified at this level of precision.

While there is justice in this observation, we wish to draw the reader's attention to the following. Both the upper and the lower bounds on α were in each case obtained from horizon complementarity; but one has no other reason to expect them to agree, even roughly. Recall that (24) and (27) are imposed directly by the geometry of the Weeks manifold [in particular, by the value taken by its inradius]. Until the recent work of Gabai et al. [51], the assertion that the Weeks manifold [volume approximately 0.9427 in curvature units] is the "smallest" compact hyperbolic space was little more than a guess: the most recent actually established previous lower bound for the volume of such a space [88] was considerably lower, at around 0.67. If a compact hyperbolic manifold with such a volume had existed, its inradius would presumably have been somewhat smaller than the inradius of the Weeks manifold [$\approx$ 0.519]. If the inradius had been [say] 0.45, then for example (24) would have been replaced by a requirement that α should be at least 3.24, which would not be easy to reconcile with (35). In short, even getting the ranges permitted by the two constraints to overlap is not trivial. Indeed, if the precision of the agreement of (24) and (35) is mere coincidence, it remains remarkable that the [pure] numbers involved are even *similar in order of magnitude.* This encourages the hope that a more sophisticated treatment might lead to a reasonable estimate for α.

Ultimately it should be possible to check this claim directly. With a more complete understanding, perhaps derived from string cosmology, of the precise nature of the inflaton [and of whatever other fields are important in the earliest Universe], one should be able to *compute* α from the structure of those fields and from the detailed geometry of the Weeks manifold. Agreement with the predictions implied by (37) or (38) could be regarded as confirmation of horizon complementarity.

In this work we have proposed that the consequences of cosmic holography can be understood in a concrete way with the aid of the Casimir effect. The underlying *reason* for the validity of the holographic point of view remains to be understood, and this is of course a major question. It seems possible, for example, that holography arises here in connection with the peculiarities of quantum field theory on negatively curved spaces [89]. Callan and Wilczek observe, for example, that negative curvature effectively confines even long-range interactions to finite domains, because of the exponential growth of surface area in hyperbolic space. This observation, and others in [89], make the "effective finiteness" of such spaces seem less surprising; but much remains to be done to extend such insights to a concrete understanding of holography and complementarity.

Acknowledgements

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